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Variational approaches to solving initial-boundary-value problems in the dynamics of linear elastic systems☆

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ABSTRACT

Problems of the controlled motion of an elastic body are considered in the linear theory. Using the method of integrodifferential relations, a family of quadratic functionals is introduced, which define the state of the elastic body, and variational formulations of the initial-boundary-value problem of dynamics are given. Euler's equations and boundary and terminal relations corresponding to them are obtained from the condition for the functionals to be stationary. It is shown that there is a relation between the proposed formulations and the Hamilton variational principle in the case of boundary-value and time-periodic problems of dynamics. A numerical algorithm is developed for finding the motions of an elastic body, based on piecewise-polynomial approximations and a criterion is proposed for estimating the quality of the approximate solutions. An example of the calculation and analysis of the forced transverse motions of a rectilinear beam with a square cross section is given for the three-dimensional model.

Hamilton's principle and the principle formulated for motions with periodic or time-boundary conditions are the most well known of the variational approaches to solving dynamic problems in the theory of elasticity (see Ref. 1). Possible generalizations of these variational principles (Heiliger-Reisner, Washizu, etc.) have also been proposed. Using a Laplace transformation other variational principles have been obtained for solving initial-boundary-value problems of the dynamics of elastic bodies; these are described together with a detailed bibliography in Ref. 2. A systematic approach to obtaining variational principles, based on a Friedrichs transformation[2], should be noted. There are also other approaches to deriving variational formulations, of which we note the semi-inverse method.[3] Among the numerical methods of solving initial-boundary-value problems in mechanics there is also the Petrov-Galerkin method.[4,5] Recently a number of papers have been published devoted to the use of the method of least squares in the mechanics of a deformable solid.[6]

In all these approaches it is assumed that certain constitutive equations, for example, the equations of motion, stress boundary conditions, etc., are satisfied approximately in numerical procedures, while the exact solution is approximated by a finite series of basis functions. When constructing a finite-dimensional approximation of the motions of an elastic body it is fairly difficult to obtain a bilateral estimate of the convergence of the numerical process. This does not enable the errors of the calculations, connected, for example, with the discretization of the problem, the rounding errors, numerical integration, etc., to be effectively monitored.

For a series of mechanical systems with distributed parameters it is possible, using the method of separation of variables, to reduce the initial problem in partial derivatives to a denumerable system of ordinary differential equations.[7] Even then, when obtaining the eigenfrequencies and an orthonormalized basis of eigenfunctions, it is necessary to use particular approaches, for example, the perturbation method (the small parameter method) in the case of slightly inhomogeneous systems.[8] An accelerated convergence method was proposed in Ref. 9, based on the classical Rayleigh–Ritz approach, which enables one, with a specified accuracy, to obtain the eigenfunctions for one-dimensional problems with arbitrarily distributed mechanical parameters. Various approaches are used for the effective modelling and control of the motions of elastic systems with distributed parameters, based, for example, on the decomposition method[10,11] or direct discretization.[12]

The purpose of the present paper is to develop, for dynamic problems of the linear theory of elasticity, the method of integrodifferential relations proposed in previous papers,[13–21] based on the integral representation of the local equations of state of a mechanical system. One of its advantages compared with the classical variational methods in dynamics is the fact that a minimum of a non-negative functional is sought. Note also that bilateral energy estimates of the quality of any admissible motion (not only optim motion) can be obtained in explicit

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form. Local estimates, which follow from the proposed integrodifferential formulation of problems of the theory of elasticity, provide one with the possibility of finding regions where the numerical solution is unreliable and effectively use different strategies to improve its quality.

1. Formulation of the problem. We will consider controlled dynamical systems with distributed inertial and elastic parameters, defined by linear partial differential equations with initial and boundary conditions. The most well-known examples of such systems, which will be discussed below or to which the approach proposed in this paper can be applied, are models describing the elastic motions of a body,[16] beams[14], strings, membranes, plates, etc. within the framework of the linear theory.

Consider an elastic body, occupying in n-dimensional space a certain bounded region Ω with a piecewise-smooth boundary γ. We will first introduce the force variables $\mathbf{p}$, σ and the geometrical variables $\mathbf{u}$, ε, which characterize the behaviour of the elastic system and depend on the time t and on the n-dimensional vector $x = \{x_1,\ldots,x_n\}$ of the space coordinates. Here $\mathbf{p}$ and $\mathbf{u}$ are vector functions of the momentum and displacement densities respectively, while σ and ε are tensors, defining the distribution of the elastic stresses and strains in time and space respectively.

In the linear theory of elasticity, the local equations of state of the medium, relating the momentum density $\mathbf{p}$ and the velocities of points of the system $\partial\mathbf{u}/\partial t$, and also the stresses σ and the strains ε, can be written in the form

$$\boldsymbol{\eta} \equiv \frac{\partial \mathbf{u}}{\partial t} - \frac{\mathbf{p}}{\rho} = 0, \quad \boldsymbol{\xi} \equiv \boldsymbol{\varepsilon} - \mathbf{C}^{-1} : \boldsymbol{\sigma} = 0 \tag{1.1}$$

Here the strain tensor ε depends linearly on the displacement vector $\mathbf{u}$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right) \tag{1.2}$$

Here the volume density of the body ρ and the modulus of elasticity tensor $\mathbf{C}$ are specified functions of the x coordinate. The components C_{ijkl} of the fourth-rank tensor $\mathbf{C}$ possess the following symmetry properties: $C_{ijkl} = C_{ijlk} = C_{klij}$. To abbreviate the notation we will introduce the vector η, which defines the relation between the displacements and the momentum density, and also the dimensionless tensor ξ, which specifies the relations of Hooke's law. In the second equation of (1.1) the colon between the tensors denotes their double scalar product (double convolution with respect to the indices). In Eq. (1.2) we have introduced the gradient operator $\nabla = (\partial/\partial x_1,\ldots,\partial/\partial x_n)^T$ in x-coordinate space.

Using relations (1.1) and (1.2), the equation of motion of an elastic body, written in displacements

$$\frac{\partial^2\mathbf{u}}{\partial t^2} - \nabla\cdot\mathbf{C} : \boldsymbol{\varepsilon} - \mathbf{f} = 0$$

can be represented, using the stress tensor σ and the momentum density vector $\mathbf{p}$, in the form

$$\frac{\partial \mathbf{p}}{\partial t} - \nabla\cdot\boldsymbol{\sigma} - \mathbf{f} = 0 \tag{1.3}$$

Here $\mathbf{f}$ is the vector of the external bulk forces, which may be either specified functions of time t and of the x coordinates, or unknown actions, if we are considering the problem of controlling the motions of an elastic body. The dot between the vectors and tensors indicates the scalar product. Since Eq. (1.3) relates the elastic and inertial bulk forces, henceforth this relation will be called the dynamic-equilibrium equation.

We will consider the case of linear boundary conditions, which are expressed in terms of the components of the displacement vector $\mathbf{u}$ and the external load vector $\mathbf{q}$ in the form

$$\alpha_k(x)u_k + \beta_k(x)q_k = \upsilon_k, \quad x\in\gamma; \quad \mathbf{q} = \boldsymbol{\sigma}\cdot\mathbf{n}, \quad k = 1,\ldots,n \tag{1.4}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the boundary of the body γ and v_k is the component of the boundary vector $\mathbf{v}$, which is either a specified function of t and x, or is controllable. The coefficients α_k and β_k depend on the x coordinates and define the type of boundary conditions. Thus, the function v_k defines the displacement u_k if the coefficients $\alpha_k = 1$ and $\beta_k = 0$ are specified on a certain part of the boundary $\gamma_u^{(k)} \in \gamma$, or defines the external load q_k if $\alpha_k = 0$ and $\beta_k = 1$ on the part of the boundary $\gamma_\sigma^{(k)} \in \gamma$. Conditions (1.4) include different forms of elastic foundation, if, for any of the indices $k = 1, \ldots, n$, a pair of coefficients α_k and β_k is simultaneously non-zero (usually $\alpha_k\beta_k > 0$).

For a complete description of the motion of an elastic body it is necessary to determine its state at the initial instant of time (without loss of generality when $t = 0$). This can be done by specifying the initial distributions of the elastic displacements $\mathbf{u}$ and the momentum density $\mathbf{p}$ as fairly continuous functions of the x coordinates:

$$\mathbf{u}(0,x) = \mathbf{u}^0(x), \quad \mathbf{p}(0,x) = \mathbf{p}^0(x), \quad x\in\Omega. \tag{1.5}$$

Relations (1.1)–(1.3) describe the stress-strain state at each internal point of the elastic body. For a solution of the initial-boundary-value problem (1.1)–(1.5) to exist, the stresses at internal points of the body must convert continuously into stresses on the boundary. Thus, the values of the displacements of internal points directly convert into boundary values. Here it is implicitly assumed that the transition of the components of the modulus of elasticity tensor $\mathbf{C}$, defined for internal points, is continuous on the boundary of the body γ.[1] On the other hand, it is necessary to bear in mind that boundary conditions (1.4) cannot be specified without taking into account the physical factors which generate these conditions. For example, a certain part of the boundary may be the interface of two or several media (both elastic and inelastic). In this case any boundary point belongs both to the body considered and to the bodies which generate these boundary conditions. Certain internal surfaces of the body Ω may also be interfaces of different materials. Any point on these surfaces simultaneously belongs to parts of the body with different mechanical properties, i.e. the modulus of elasticity tensor $\mathbf{C}$ and the function of the density ρ on these surfaces, generally speaking, are not defined and, consequently, relations (1.1) are uncertain.

In order to take this kind of uncertainty into account, variational approaches are used in static problems of the linear theory of elasticity, for example, the principles of the minimum of the potential and complementary energy.[1] In dynamics, variational principles are only formulated for boundary-value problems.

We will give two classical examples which will be required later.

When the displacements $\mathbf{u}$ are specified at the beginning ($t=0$) and the end ($t=t_f$) of the motion, the kinetic energy function T and the total potential energy function Π exist, which are expressed in terms of the displacements, and the surface loads $\mathbf{q}$ do not change for variations $\delta\mathbf{u}$, the principle that the Hamiltonian is stationary follows from the principle of virtual work

$$
\begin{aligned}
&\delta H = 0, \quad H = \int_0^{t_f} (T - \Pi)\,dt \\
&T = \int_\Omega K\,d\Omega, \quad \Pi = \int_\Omega [A - \mathbf{f}\cdot\mathbf{u}]\,d\Omega - \sum_{i=1}^{n} \int_{\gamma_\sigma^{(i)}} u_i \upsilon_i\,d\gamma \\
&K = \frac{\rho}{2}\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t}, \quad A = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon}; \quad u_k = \upsilon_k, \quad x \in \gamma_u^{(k)}, \quad k = 1,\dots,n \\
&\mathbf{u}(0,x) = \mathbf{u}^0, \quad \mathbf{u}(t_f,x) = \mathbf{u}^f
\end{aligned}
\tag{1.6}
$$

In this problem relations (1.1) are satisfied, and the components u_k of the displacement vector $\mathbf{u}$ must strictly satisfy boundary condition (1.4) on the parts of the boundary $\gamma_u^{(k)} \subset \gamma$, where $\alpha_k = 1$ and $\beta_k = 0$. The equation of dynamic equilibrium (1.3) and the boundary conditions in the stresses on the parts of the boundary $\gamma_\sigma^{(k)} = \gamma\backslash\gamma_u^{(k)}$, where the coefficients $\alpha_k = 0$ and $\beta_k = 1$ are specified, appear in the conditions for the Hamiltonian H, defined in system (1.6), to be stationary.

If the initial and final distributions of the momentum density $\mathbf{p}$ are specified, functions of the complementary kinetic energy T_c and the total complementary energy Π_c exist, expressed respectively in terms of the momentum density $\mathbf{p}$ and the stress σ, and for variations of the equilibrium fields $\delta\mathbf{p}$ and $\delta\sigma$ the boundary displacements do not change, we can formulate the principle for the complementary energy to be stationary

$$
\begin{aligned}
&\delta H_c = 0, \quad H_c = \int_0^{t_f} (T_c - \Pi_c)\,dt \\
&T_c = \int_\Omega K_c\,d\Omega, \quad \Pi_c = \int_\Omega A_c\,d\Omega - \sum_{i=1}^{n} \int_{\gamma_u^{(i)}} q_i \upsilon_i\,d\gamma \\
&K_c = \frac{\mathbf{p}\cdot\mathbf{p}}{2\rho}, \quad A_c = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{C}^{-1} : \boldsymbol{\sigma}; \quad q_k = \upsilon_k, \quad x \in \gamma_\sigma^{(k)}, \quad k = 1,\dots,n \\
&\mathbf{p}(0,x) = \mathbf{p}^0, \quad \mathbf{p}(t_f,x) = \mathbf{p}^f
\end{aligned}
\tag{1.7}
$$

According to this principle, the equilibrium fields of the momentum density $\mathbf{p}$ and the stresses σ satisfy relations (1.1), while the components of the tensor σ strictly satisfy the boundary conditions in stresses on the parts of the boundary $\gamma_\sigma^{(k)}$. Kinematic equation (1.2) and the boundary conditions specified in displacements on the parts of the boundary $\gamma_u^{(k)}$ are equivalent to the conditions for the functional H_c, complementary to the Hamiltonian, to be stationary.

Note that variational principles (1.6) and (1.7) are formulated for dynamic boundary-value problems of the linear theory of elasticity and, generally speaking, cannot be used in this form to solve initial-boundary-value problem (1.1)–(1.5).

2. The method of integrodifferential relations. In the proposed approach, unlike the classical formulations, instead of local equations of state of the elastic body (1.1), we consider a single integral relation, connecting the vector of the momentum density $\mathbf{p}$ and the velocity vector $\partial\mathbf{u}/\partial t$, and also the stress tensor σ and the strain tensor ε.

The following integrodifferential formulation of the initial-boundary-value problem of the motion of an elastic body (1.1)–(1.5) was given in Ref. 14: it is required to obtain unknown fields of the displacements $\mathbf{u}^*$, the stresses σ^* and the momentum density $\mathbf{p}^*$, which satisfy the following integral relation

$$
\begin{aligned}
&\Phi_+ = \int_0^{t_f}\int_\Omega \varphi_+(\mathbf{u},\boldsymbol{\sigma},\mathbf{p})\,d\Omega\,dt = 0 \\
&\varphi_+ = \frac{1}{2}[\rho\boldsymbol{\eta}\cdot\boldsymbol{\eta} + \boldsymbol{\xi} : \mathbf{C} : \boldsymbol{\xi}], \quad \boldsymbol{\eta} = \frac{\partial\mathbf{u}}{\partial t} - \frac{\mathbf{p}}{\rho}, \quad \boldsymbol{\xi} = \boldsymbol{\varepsilon} - \mathbf{C}^{-1} : \boldsymbol{\sigma}
\end{aligned}
\tag{2.1}
$$

with strict satisfaction of kinematic equation (1.2), the dynamic equilibrium equation (1.3), and the boundary and initial conditions (1.4) and (1.5).

Note that the integrand function φ_+, defined in system (2.1), has the dimension of energy density and is non-negative. It follows from the last property that the integral Φ_+ is non-negative for arbitrary functions $\mathbf{u}$, σ and $\mathbf{p}$, This property enables us to reduce integrodifferential problem (1.2)–(1.5), (2.1) to a minimization formulation, namely, it is required to obtain the admissible functions $\mathbf{u}^*$, σ^* and $\mathbf{p}^*$ which give a minimum (zero) value to the functional Φ_+:

$$\Phi_+(\mathbf{u}^*, \boldsymbol{\sigma}^*, \mathbf{p}^*) = \min_{\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p}} \Phi_+(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p}) = 0 \tag{2.2}$$

with strict satisfaction of constraints (1.2)–(1.5).

We will denote the real and arbitrarily chosen admissible displacements, stresses, and momenta by $\mathbf{u}^*$, σ^*, $\mathbf{p}^*$ and $\mathbf{u}$, σ and $\mathbf{p}$ respectively and put

$$\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}, \quad \mathbf{p} = \mathbf{p}^* + \delta\mathbf{p}$$

Then

$$\Phi_+(\mathbf{u},\boldsymbol{\sigma},\mathbf{p}) = \delta_{\mathbf{u}}\Phi_+ + \delta_{\boldsymbol{\sigma}}\Phi_+ + \delta_{\mathbf{p}}\Phi_+ + \delta^2\Phi_+$$

where $\delta_u\Phi_+$, $\delta_\sigma\Phi_+$, $\delta_p\Phi_+$ are the first variations with respect to the variables $\mathbf{u}$, σ and $\mathbf{p}$, while $\delta^2\Phi_+ \geq 0$ is the second variation of the functional Φ_+.

It was shown in Refs. 13, 17 and 20 that it is possible to construct families of equivalent variational formulations with different minimised functionals for static problems of the linear theory of elasticity. Similarly, in dynamic problems one can introduce the following parametric family of quadratic functionals

$$\Phi = \int_0^{t_f}\int_\Omega \varphi d\Omega dt, \quad \varphi = \alpha\rho\boldsymbol{\eta}\cdot\boldsymbol{\eta} + \beta\boldsymbol{\xi} : \mathbf{C} : \boldsymbol{\xi}, \quad \alpha^2 + \beta^2 = \frac{1}{2}, \quad \alpha \geq 0 \tag{2.3}$$

for which, when constraints (1.2)–(1.5) are satisfied, the stationary conditions are equivalent to constitutive relations (1.1) of the state of the elastic body. In expressions (2.3) for the functional Φ the quantities α and β are real constants. When $\alpha = \beta = ½$ the integral of Φ is identical with the non-negative functional Φ_+, defined by relations (2.1). If $\beta>0$ and $\alpha \neq 0$, for each functional $\Phi[\beta]$ there is a problem of conditional minimization, similar to problem (1.2)–(1.5), (2.2).

For arbitrary non-zero values of α and β ($\alpha\beta \neq 0$) we can define a general variational formulation of the initial-boundary-value problem of linear elasticity, namely, it is required to obtain the unknown functions $\mathbf{u}^*$, σ^*, $\mathbf{p}^*$, which, when constraints (1.2)–(1.5) are satisfied, give stationary values to the functional Φ

$$\delta_{\mathbf{u}}\Phi + \delta_{\boldsymbol{\sigma}}\Phi + \delta_{\mathbf{p}}\Phi = 0 \tag{2.4}$$

After integration by parts, taking constraints (1.2)–(1.5) into account, we obtain expressions for the first variations

$$\begin{aligned}\frac{\delta_{\mathbf{u}}\Phi}{2} &= -\int_0^{t_f}\int_\Omega\left[\alpha\rho\frac{\partial\boldsymbol{\eta}}{\partial t} + \beta\nabla\cdot(\mathbf{C}:\boldsymbol{\xi})\right]\cdot\delta\mathbf{u}d\Omega dt + \\ &+\alpha\int_\Omega[\rho\boldsymbol{\eta}\cdot\delta\mathbf{u}]_{t=0}^{t=t_f}d\Omega + \beta\int_0^{t_f}\int_\gamma\mathbf{n}\cdot(\mathbf{C}:\boldsymbol{\xi})\cdot\delta\mathbf{u}d\gamma dt \\ \delta_{\boldsymbol{\sigma}}\Phi &= -2\beta\int_0^{t_f}\int_\Omega\boldsymbol{\xi}:\delta\boldsymbol{\sigma}d\Omega dt, \quad \delta_{\mathbf{p}}\Phi = -2\alpha\int_0^{t_f}\int_\Omega\boldsymbol{\eta}\cdot\delta\mathbf{p}d\Omega dt\end{aligned} \tag{2.5}$$

It can be seen from expressions (2.5) that the first variations of the functional Φ are equal to zero in any admissible variations of the functions $\delta\mathbf{u}$, $\delta\sigma$, $\delta\mathbf{p}$, if relations (1.1) are satisfied.

We will obtain, in explicit form, the necessary conditions for the functional Φ to be stationary (Euler's equations). To do this it is necessary to take into account the relation between the stress tensor σ and the momentum density vector $\mathbf{p}$, which is defined by dynamic equilibrium equation (1.3) and, of course, the relation between their variations $\delta\mathbf{p}$ and $\delta\sigma$

$$\frac{\partial\delta\mathbf{p}}{\partial t} = \nabla\cdot\delta\boldsymbol{\sigma} \tag{2.6}$$

Introducing the auxiliary vector

$$\boldsymbol{\zeta} = -\int_0^t\boldsymbol{\eta}dt \tag{2.7}$$

we obtain, taking constraint (2.6) into account, the following expressions for the first variations $\delta\Phi$

$$\begin{aligned}
&\delta_{\mathbf{u}}\Phi + \delta_{\sigma}\Phi = 0\\
&\frac{\delta_{\mathbf{u}}\Phi}{2} = \int_0^{t_f}\int_\Omega\left[\alpha\rho\frac{\partial^2\zeta}{\partial t^2} - \beta\nabla\cdot(\mathbf{C}:\xi)\right]\cdot\delta\mathbf{u}d\Omega dt - \alpha\int_\Omega\left[\rho\frac{\partial\zeta}{\partial t}\cdot\delta\mathbf{u}\right]_{t=0}^{t=t_f}d\Omega + \beta\int_0^{t_f}\int_\gamma\mathbf{n}\cdot(\mathbf{C}:\xi)\cdot\delta\mathbf{u}d\gamma dt\\
&\delta_\sigma\Phi = 2\int_0^{t_f}\int_\Omega(\alpha\varepsilon_\zeta - \beta\xi):\delta\sigma d\Omega dt + \alpha\int_\Omega[\zeta\cdot\delta\mathbf{p}]_{t=0}^{t=t_f}d\Omega - \alpha\int_0^{t_f}\int_\gamma\zeta\cdot\delta\sigma\cdot\mathbf{n}d\gamma dt\\
&\varepsilon_\zeta = \frac{1}{2}(\nabla\zeta + \nabla\zeta^T)
\end{aligned}\tag{2.8}$$

Taking into account conditions (1.4) and (1.5), imposed on the boundary and initial variations of the unknown functions $\delta\mathbf{u}$, $\delta\sigma$, $\delta\mathbf{p}$, we can write Euler's equations with the corresponding boundary and terminal relations

$$\begin{aligned}
&\rho\frac{\partial^2\zeta}{\partial t^2} - \nabla\cdot(\mathbf{C}:\varepsilon_\zeta) = 0, \quad \alpha\varepsilon_\zeta + \beta\xi = 0\\
&\alpha_k\zeta_k + \beta_k r_k = 0, \quad x\in\gamma; \quad \mathbf{r} = (\mathbf{C}:\varepsilon_\zeta)\cdot\mathbf{n}, \quad k = 1,\ldots,n\\
&\zeta|_{t=t_f} = \left.\frac{\partial\zeta}{\partial t}\right|_{t=t_f} = 0
\end{aligned}\tag{2.9}$$

With respect to the unknown vector ζ, system (2.9) is a terminal-boundary-value problem of the linear theory of elasticity with a unique zero solution, from which the conditions $\xi = 0$ and $\eta = 0$ follow directly. In other words, if the solution $\mathbf{u}^*$, σ^* and $\mathbf{p}^*$ of problem (1.2)–(1.5), (2.4) exists, system (2.9) together with constraints (1.2)–(1.5) is equivalent to the initial dynamic problem of the linear theory of elasticity (1.1)–(1.5).

3. Analysis of the variational formulations. We will consider in more detail one functional of the family (2.3) when $\alpha = -\beta = 1/2$ ($\Phi_- = \Phi|_{\beta=-1/2}$). We will represent it in the form

$$\begin{aligned}
&\Phi_- = \int_0^{t_f}\int_\Omega\varphi_-(\mathbf{u},\sigma,\mathbf{p})d\Omega dt = \Theta_u + \Theta_\sigma - 2\Theta\\
&\varphi_- = \frac{1}{2}[\rho\eta\cdot\eta - \xi:\mathbf{C}:\xi], \quad \Theta = \frac{1}{2}\int_0^{t_f}\int_\Omega\left[\mathbf{p}\cdot\frac{\partial\mathbf{u}}{\partial t} - \sigma:\varepsilon\right]d\Omega dt\\
&\Theta_u = \frac{1}{2}\int_0^{t_f}\int_\Omega\left[\rho\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t} - \varepsilon:\mathbf{C}:\varepsilon\right]d\Omega dt, \quad \Theta_\sigma = \frac{1}{2}\int_0^{t_f}\int_\Omega\left[\rho^{-1}\mathbf{p}\cdot\mathbf{p} - \sigma:\mathbf{C}^{-1}:\sigma\right]d\Omega dt
\end{aligned}\tag{3.1}$$

Note that the integral Θ_u depends only on the displacements $\mathbf{u}$, while the vector-function $\mathbf{u}$ does not occur in the expression for Θ_σ. The bilinear functional Θ, in turn, is explicitly independent of the elastic and inertial properties of the solid, and after integration by parts it can be reduced to the form

$$2\Theta = \int_\Omega[\mathbf{p}\cdot\mathbf{u}]_{t=0}^{t=t_f}d\Omega - \int_0^{t_f}\int_\Omega\mathbf{f}\cdot\mathbf{u}d\Omega dt - \int_0^{t_f}\int_\gamma\mathbf{q}\cdot\mathbf{u}d\gamma dt \tag{3.2}$$

As in problems (1.6) and (1.7), we will consider the case when, on the part $\gamma_\sigma^{(k)}$ of the boundary γ, the component q_k of the load vector $\mathbf{q}$ $\alpha_k = 0$, $\beta_k = 1$ (in Eq. (1.4)) is specified in terms of the function v_k, whereas on the remaining part of the boundary $\gamma_u^{(k)} \subset \gamma$ the component u_k of the displacement vector $\mathbf{u}$ ($\alpha_k = 1$, $\beta_k = 0$) is specified. We emphasise that the parts of the boundary $\gamma_\sigma^{(k)}$, $\gamma_u^{(k)}$ satisfy the following conditions

$$\overline{\gamma_u^{(k)}\cup\gamma_\sigma^{(k)}} = \gamma, \quad \gamma_u^{(k)}\cap\gamma_\sigma^{(k)} = \varnothing, \quad k = 1,\ldots,n$$

i.e., there are no mixed boundary relations connecting the components of the displacement vector and the strain tensor. Then, after substituting expression (3.2), the functional Φ_- can be rewritten as follows:

$$\Phi_- = \Theta_u + \int_0^{t_f}\int_\Omega\mathbf{f}\cdot\mathbf{u}d\Omega dt + \sum_{i=1}^n\int_0^{t_f}\int_{\gamma_\sigma^{(i)}}\upsilon_i u_i d\gamma dt + \Theta_\sigma + \sum_{i=1}^n\int_0^{t_f}\int_{\gamma_u^{(i)}}q_i\upsilon_i d\gamma dt - \int_\Omega[\mathbf{p}\cdot\mathbf{u}]_{t=0}^{t=t_f}d\Omega \tag{3.3}$$

Table 1

Condition A	Condition B	Ξ_u	Ξ_p
$\mathbf{u}(0,x)=\mathbf{u}^0(x)$	$\mathbf{u}(t_f,x)=\mathbf{u}^f(x)$	0	$\int_\Omega [\mathbf{u}^f\cdot\mathbf{p}\vert_{t=t_f}-\mathbf{u}^0\cdot\mathbf{p}\vert_{t=0}]d\Omega$
$\mathbf{p}(0,x)=\mathbf{p}^0(x)$	$\mathbf{p}(t_f,x)=\mathbf{p}^f(x)$	$\int_\Omega [\mathbf{p}^f\cdot\mathbf{u}\vert_{t=t_f}-\mathbf{p}^0\cdot\mathbf{u}\vert_{t=0}]d\Omega$	0
$\mathbf{u}(0,x)=\mathbf{u}^0(x)$	$\mathbf{p}(t_f,x)=\mathbf{p}^f(x)$	$\int_\Omega [\mathbf{p}^f\cdot\mathbf{u}\vert_{t=t_f}]d\Omega$	$-\int_\Omega [\mathbf{u}^0\cdot\mathbf{p}\vert_{t=0}]d\Omega$
$\mathbf{p}(0,x)=\mathbf{p}^0(x)$	$\mathbf{u}(t_f,x)=\mathbf{u}^f(x)$	$-\int_\Omega [\mathbf{p}^0\cdot\mathbf{u}\vert_{t=0}]d\Omega$	$\int_\Omega [\mathbf{u}^f\cdot\mathbf{p}\vert_{t=t_f}]d\Omega$
$\mathbf{u}(0,x)=\mathbf{u}^0(t_f,x)$	$\mathbf{p}(0,x)=\mathbf{p}(t_f,x)$	0	0

If, at the initial and final instants of time, we are given the distribution in the body of either the displacements or the momentum density, the integral Φ_- splits into two independent parts

$$\Phi_- = H_u + H_\sigma$$
$$H_u = \Theta_u + \int_0^{t_f}\int_\Omega \mathbf{f}\cdot\mathbf{u}d\Omega dt + \sum_{i=1}^{3}\int_0^{t_f}\int_{\gamma_\sigma^{(i)}} \upsilon_i u_i d\gamma_\sigma^{(i)}dt + \Xi_u$$
$$H_\sigma = \Theta_\sigma + \sum_{i=1}^{3}\int_0^{t_f}\int_{\gamma_u^{(i)}} q_i\upsilon_i d\gamma_u^{(i)}dt + \Xi_p \tag{3.4}$$

The functional H_u depends only on the displacement function $\mathbf{u}$, while H_σ depends on the stress tensor σ and the momentum density vector $\mathbf{p}$, and Ξ_u and Ξ_p are integrals over the region Ω, which depend on the functions $\mathbf{u}$ and $\mathbf{p}$ respectively. In Table 1 we show the time boundary conditions (A, B) and explicit expressions for the integrals Ξ_u and Ξ_p for different types of dynamic problems. The first four rows correspond to boundary-value problems, while the last row corresponds to a time-periodic problem.

According to the first equality of (2.8), when taking constraints (1.3) and (1.4) and the time boundary conditions shown in the table into account, the stationary conditions for the functional Φ_- can be represented in the form

$$\delta\Phi_- = \delta_\mathbf{u}H_u + \delta_\sigma H_\sigma = 0 \tag{3.5}$$

Consequently, the boundary-value problem considered can be split into two independent subproblems.

Problem 1. It is required to find the admissible field of displacements $\mathbf{u}^*$, which satisfies conditions (1.4) on the parts of the boundary $\gamma_u^{(k)}(k=1,\dots,n)$, and also the initial and final conditions from the table, specified in displacements, and which delivers a stationary value to the functional H_u:

$$\delta_\mathbf{u}H_u = 0 \tag{3.6}$$

Problem 2. It is required to find the equilibrium fields of the stresses σ^* and the momentum density $\mathbf{p}^*$, which satisfy conditions (1.4) on the parts of the boundary $\gamma_\sigma^{(k)}(k=1,\dots,n)$, and also the initial and final conditions from the table, specified in momenta, and which yield a stationary value to the functional H_σ:

$$\delta_\sigma H_\sigma = 0 \tag{3.7}$$

It is important to note that if the displacements (the first row of the table) are specified at the initial and final instants of time, Problem 1 agrees with the Hamilton stationarity principle (1.6) ($H_u = H$). If the distribution of the momentum density vector (the second row of the table) is determined at the beginning and end of the process, Problem 2 corresponds completely to the complementary Hamilton principle (1.7) ($H_\sigma = H_c$). Both these classical principles follow from Problems 1 and 2 in the periodic case (the last row of the table).

Unlike variational principles (1.6) and (1.7), which are formulated for dynamic boundary-value problems, in the proposed approach the functional Φ can also be used to solve initial-boundary-value problem (1.1)–(1.5). When initial conditions (1.5) are specified the last term in relations (3.3) depends on the scalar product of the vectors $\mathbf{u}$ and $\mathbf{p}$, taken at the final instant of time. This does not enable the functional Φ_- to be split into two independent parts. The separation into two subproblems (3.6) and (3.7) is even impossible for the boundary-value problems defined in the table if the mixed conditions ($\alpha_k\beta_k \neq 0$) are specified in some part of the boundary γ of the region Ω. In this case the variational problem (2.4) with the corresponding constraints must be solved simultaneously in displacements, stresses and momenta.

4. Variational formulation of the initial-boundary-value problem in displacements and stresses. When solving dynamic problems of the linear theory of elasticity in the proposed variational formulation (1.2)–(1.5), (2.4) one has to deal with quite a large number of unknown quantities. Thus, for a three-dimensional problem ($n = 3$) it is necessary to determine six components of the displacement vectors $\mathbf{u}$ and the momentum density $\mathbf{p}$ and six components of the stress tensor σ. Even after solving vector equilibrium equation (1.3), nine independent functions remain in a functional. This may considerably limit the applicability of numerical approaches. To increase the effectiveness of

computational algorithms, it is possible to use special variational formulations in which part of the equations of states (1.1) is regarded as additional constraints on the admissible states of the solid.

We will consider a non-negative functional from the family (2.3), namely, the integral Φ with $\alpha = 0$ and $\beta = 1/\sqrt{2}$:

$$\Phi_0 \equiv \frac{1}{\sqrt{2}}\Phi\Big|_{\alpha=0} = \int_0^{t_f}\int_\Omega \varphi_0(\mathbf{u},\boldsymbol{\sigma})d\Omega dt$$
$$\varphi_0 = \frac{1}{2}\xi : \mathbf{C} : \xi, \quad \xi = \boldsymbol{\varepsilon} - \mathbf{C}^{-1} : \boldsymbol{\sigma} \tag{4.1}$$

For the correct formulation of the minimization problem, corresponding to functional Φ_0, the relation of the velocity and momentum density $\eta = 0$ in Eqs (1.1) must be regarded as a local constraint on the displacement and momentum fields. Then, the variational formulation sounds as follows: it is required to obtain such displacement and stress fields $\mathbf{u}^*$ and σ^*, which give the functional Φ_0 a minimum value for the specified constraints

$$\Phi_0(\mathbf{u}^*,\boldsymbol{\sigma}^*) = \min_{\mathbf{u},\boldsymbol{\sigma}} \Phi_0(\mathbf{u},\boldsymbol{\sigma}) = 0$$
$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}$$
$$\alpha_k(x)u_k + \beta_k(x)q_k = \upsilon_k, \quad x \in \gamma; \quad \mathbf{q} = \boldsymbol{\sigma}\cdot\mathbf{n}, \quad k = 1,\dots,n$$
$$\mathbf{u}(0,x) = \mathbf{u}^0(x), \quad \frac{\partial\mathbf{u}}{\partial t}(0,x) = \frac{\mathbf{p}^0(x)}{\rho} \tag{4.2}$$

Just as in Section 3, we will denote the actual admissible displacements and stresses by $\mathbf{u}^*$ and σ^* and $\mathbf{u}$ and σ respectively. We will introduce the notation

$$\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}$$

and we will write the variations of the functional

$$\Phi_0(\mathbf{u},\boldsymbol{\sigma}) = \delta_{\mathbf{u}}\Phi_0 + \delta_{\boldsymbol{\sigma}}\Phi_0 + \delta^2\Phi_0$$

The second variations are non-negative by virtue of the fact that the integrand function φ_0 is quadratic in the variations $\delta\mathbf{u}$ and $\delta\sigma$ ($\delta^2\Phi_0 \geq 0$). The first variations of the functional Φ_0 are equal to zero for arbitrary $\delta\mathbf{u}$ and $\delta\sigma$ if the equality $\xi = 0$ is satisfied.

In problem (4.2) the displacement vector $\mathbf{u}$ and the stress tensor σ, and also their variations $\delta\mathbf{u}$ and $\delta\sigma$, are related, in addition to possible mixed boundary conditions, by the dynamic equilibrium relations

$$\frac{\partial^2\delta\mathbf{u}}{\partial t^2} = \nabla\cdot\delta\boldsymbol{\sigma}. \tag{4.3}$$

To determine the conditions for the functional Φ_0 to be stationary with corresponding constraints, we will introduce into consideration the auxiliary tensor χ, which defines the equilibrium fields of the displacements and stresses so that

$$\mathbf{u} = \nabla\cdot\chi + \int_0^{t_f}\int_0^{t_f}\frac{\mathbf{f}}{\rho}dtdt, \quad \boldsymbol{\sigma} = \rho\frac{\partial^2\chi}{\partial t^2} \tag{4.4}$$

After a number of substitutions and integrations by parts, the first variations of the functional Φ_0, expressed in terms of the tensors ξ and χ, have the form

$$\delta_\chi\Phi_0 = \int_0^{t_f}\int_\Omega\left[\Delta_\xi - \rho\frac{\partial^2\xi}{\partial t^2}\right] : \delta\chi d\Omega dt + \int_\Omega \rho\left[\frac{\partial\xi}{\partial t} : \delta\chi - \xi : \frac{\partial\delta\chi}{\partial t}\right]_{t=0}^{t=t_f} d\Omega +$$
$$+ \int_0^{t_f}\int_\gamma[(\mathbf{n}\cdot\mathbf{C} : \xi)\cdot(\nabla\cdot\delta\chi) - (\nabla\cdot\mathbf{C} : \xi)\cdot(\delta\chi\cdot\mathbf{n})]d\gamma dt = 0$$
$$\Delta_\xi = \frac{1}{2}\left[\nabla(\nabla\cdot\mathbf{C} : \xi) + \nabla(\nabla\cdot\mathbf{C} : \xi)^T\right] \tag{4.5}$$

The system of Euler's equations and the corresponding boundary and terminal conditions can be obtained from conditions of stationarity (4.5), taking into account the boundary and initial conditions, imposed on the variations of the tensor χ,

$$\rho\frac{\partial^2\xi}{\partial t^2} = \frac{1}{2}\left[\nabla(\nabla\cdot\mathbf{C}:\xi) + \nabla(\nabla\cdot\mathbf{C}:\xi)^T\right]$$
$$\alpha_k(\nabla\cdot\mathbf{C}:\xi)_k + \beta_k(\mathbf{n}\cdot\mathbf{C}:\xi)_k = 0, \quad x\in\gamma, \quad k = 1,\ldots,n$$
$$\xi\big|_{t=t_f} = \frac{\partial\xi}{\partial t}\bigg|_{t=t_f} = 0 \tag{4.6}$$

If the solution of the original dynamic problem exists, it can be shown that the terminal-boundary-value problem (4.6) in the unknown tensor ξ has only the trivial solution $\xi = 0$. This indicates that the stationarity conditions (4.5) together with constraints (4.2) are equivalent to the complete system of equations (1.1)–(1.5), describing the motions of an elastic solid in the linear theory of elasticity.

5. Numerical algorithm and estimates of the quality of the solution. We will consider one of the possible algorithms for the numerical construction of the solution of minimization problem (4.2) and illustrate it using the example of the motions of a homogeneous isotropic elastic solid (n = 3). We will confine ourselves to the case when there are no bulk forces $\mathbf{f} = 0$.

The proposed algorithm can be described in general form as follows. Initially we specify the integer number N and choose approximations $\tilde{\mathbf{u}}$, $\tilde{\boldsymbol{\sigma}}$ of the solution $\mathbf{u}^*$ and σ^* of problem (4.2) in the form of finite sums

$$\tilde{\mathbf{u}} = \sum_{k=1}^{N}\mathbf{u}^{(k)}\psi_k(t,x), \quad \tilde{\boldsymbol{\sigma}} = \sum_{k=1}^{N}\boldsymbol{\sigma}^{(k)}\psi_k(t,x) \tag{5.1}$$

where $\{\psi_k(t,x),\ k = 1,2,\ldots\}$ is a certain complete denumerable system of linearly independent functions, and $\mathbf{u}^{(k)}$ and $\sigma^{(k)}$ are unknown real coefficients, written in vector and tensor form. The basis functions ψ_k are chosen so that the approximations $\tilde{\mathbf{u}}$, $\tilde{\boldsymbol{\sigma}}$ can exactly satisfy the dynamic-equilibrium equation, the boundary and initial conditions defined in problem (4.2) for a certain set of constants $\mathbf{u}^{(k)}$, $\sigma^{(k)}$. It directly follows from this that the boundary vector $\mathbf{v}(t, x)$, $x\in\gamma$ and functions of the initial distribution of the displacements $\mathbf{u}^0(x)$ and momenta $\mathbf{p}^0(x)$ must be represented in the form

$$\mathbf{v} = \sum_{k=1}^{N_v}\mathbf{v}^{(k)}\psi_k\big|_{x\in\gamma}, \quad \mathbf{u}^0 = \sum_{k=1}^{N_u}\mathbf{u}^{(0k)}\psi_k(0,x), \quad \mathbf{p}^0 = \sum_{k=1}^{N_p}\mathbf{p}^{(0k)}\psi_k(0,x)$$

where $\mathbf{v}^{(k)}$, $\mathbf{u}^{(0k)}$ and $\mathbf{p}^{(0k)}$ are fixed coefficients.

In the next step the part of the parameters $\mathbf{u}^{(k)}$, $\sigma^{(k)}$ is determined so that the approximations (5.1) satisfy all conditions (4.2). The admissible approximations $\tilde{\mathbf{u}}$, $\tilde{\boldsymbol{\sigma}}$ obtained are substituted into functional Φ_0. The problem of the conditional minimization of the functional Φ_0 with respect to the unknowns $\mathbf{u}$ and σ becomes a problem of unconditional minimization of the function $\Phi_0(w)$, where $w = \{u_i^{(k)}, \sigma_{ij}^{(k)}\}$ is the set of all independent parameters which remain after constraints (4.2) are satisfied. Since the function Φ_0 is quadratic in w, to obtain the optimum set of parameters w^* and its minimum value $\Phi_0^* = \Phi_0(w^*) \geq 0$ it is necessary to solve the linear system of algebraic equations $\partial\Phi_0(w)/\partial w = 0$.

Note that the functional Φ_0 reaches its absolute zero value only for the exact solution $\mathbf{u}^*$, σ^*. In order to estimate the quality of the approximate solution $\tilde{\mathbf{u}}^* = \tilde{\mathbf{u}}(t, x, w^*)$, $\tilde{\sigma}^* = \tilde{\sigma}(t, x, w^*)$ obtained, we can use the following criterion

$$\Delta = \frac{\Phi_0^*}{\tilde{\Psi}^*} < \delta; \quad \tilde{\Psi}^* = \int_0^{t_f}\tilde{W}^*dt, \quad \tilde{W}^*(t) = \int_\Omega\psi(\tilde{\mathbf{u}}^*)d\Omega, \quad \psi = \frac{1}{2}\left(\rho\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\varepsilon}:\mathbf{C}:\boldsymbol{\varepsilon}\right)$$
$$\tilde{\varphi}_0^*(t,x) = \frac{1}{2}\left[\tilde{\xi}^*:\mathbf{C}:\tilde{\xi}^*\right]; \quad \tilde{\xi}^* = \tilde{\boldsymbol{\varepsilon}}^* - \mathbf{C}^{-1}:\tilde{\boldsymbol{\sigma}}^*, \quad \tilde{\boldsymbol{\varepsilon}}^* = \frac{1}{2}\left[\nabla\tilde{\mathbf{u}}^* + (\nabla\tilde{\mathbf{u}}^*)^T\right]$$
$$\dot{\tilde{W}}^* = \int_\gamma\tilde{\mathbf{q}}^*\cdot\frac{\partial\tilde{\mathbf{u}}^*}{\partial t}d\gamma + \dot{W}_0; \quad \dot{W}_0 = \int_\Omega\tilde{\xi}^*:\mathbf{C}:\frac{\partial\tilde{\boldsymbol{\varepsilon}}^*}{\partial t}d\Omega, \quad \tilde{\mathbf{q}}^* = \mathbf{n}\tilde{\boldsymbol{\sigma}}^* \tag{5.2}$$

Here $\delta \ll 1$ is a positive dimensionless quantity, which defines the quality of the solution, $\tilde{\Psi}^* > 0$ is the approximate value of the integral of the total mechanical energy W, stored by the elastic solid during the motion, and ψ is the volume density of this energy. The quantity Δ can be regarded as the integral relative error of the approximate solution, whereas the function $\tilde{\varphi}_0^*$ of time t and coordinate x can be considered as the distribution of the local energy error. The mechanical energy function $\tilde{W}^*$ includes the term W_0, which determines, in explicit form, the difference between the work done by the external forces on the approximate motions $\tilde{\mathbf{u}}^*$, $\tilde{\boldsymbol{\sigma}}$ and the energy $\tilde{W}^*(t) - \tilde{W}^*(0)$ stored by the body. It follows from the expression for the derivative with respect to time of the energy W_0 in the penultimate formula of (5.2) that the value of the energy imbalance is directly related to the value of the relative integral error Δ.

6. Controlled motions of a three-dimensional elastic solid. We will consider, as an example, the motions of an elastic solid having, in the undeformed state, the shape of an elongated rectangular parallelepiped (Fig. 1). It is assumed that the solid is made of a homogeneous isotropic material with modulus of elasticity E and Poisson's ratio ν and volume density ρ. The following geometric parameters are given: $\Omega = \{x_1, x_2, x_3\colon 0 < x_1 < L, |x_2| < a, |x_3| < a\}$, where L is the length of the solid and 2a is the size of the cross section. The final instant of time t_f is fixed. We will consider the case when, at the initial instant of time $t = 0$, there are no elastic displacements in the solid and all points are at rest. We will chose as the control the time-polynomial of the translational displacement of the face $x_1 = 0$ as a solid whole along the Ox_3, where O is the origin of coordinates. We will assume that no external bulk forces act on the solid.

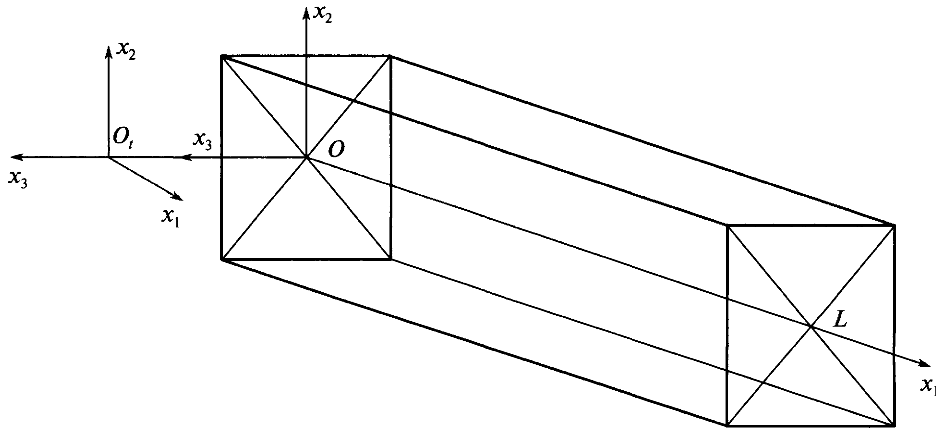

Fig. 1.

For this problem the conditions on the boundary can be written in the form

$$\begin{aligned}&\sigma_{12}\big|_{x_2=\pm a}=\sigma_{22}\big|_{x_2=\pm a}=\sigma_{23}\big|_{x_2=\pm a}=\sigma_{13}\big|_{x_3=\pm a}=\sigma_{23}\big|_{x_3=\pm a}=\sigma_{33}\big|_{x_3=\pm a}=0\\&\sigma_{11}\big|_{x_1=L}=\sigma_{12}\big|_{x_1=L}=\sigma_{13}\big|_{x_1=L}=0,\quad u_1\big|_{x_1=0}=u_2\big|_{x_1=0}=0,\quad u_3\big|_{x_1=0}=\upsilon_3\end{aligned}\tag{6.1}$$

The component υ_3 of the boundary function $\mathbf{v}$ from Eqs (1.4) is defined in the class of polynomials as a function of time with a fixed degree N_υ

$$\upsilon_3=\sum_{k=0}^{N_\upsilon}v_3^{(k)}t^k\tag{6.2}$$

where $\upsilon_3^{(k)}$ are specified constants. The initial conditions (1.5) have zero right-hand sides

$$\mathbf{u}^0(x)=\mathbf{p}^0(x)=0\tag{6.3}$$

Polynomial approximations for constructing and optimizing the longitudinal motions of an elastic solid of the same shape and an infinitely thin rod were used earlier in Ref. 18, while piecewise-polynomial splines were proposed in Ref. 21 to model the transverse motions of an Euler–Bernoulli beam. In these papers the integral Φ_+ or its analog was considered as the minimized functional for the beam model. In this paper we use a variational formation of the problem in the form (4.2).

7. Approximations of the displacement and stress fields. We will briefly write the finite-dimensional approximations employed and their properties. We will choose the degree of approximation M_p and the following polynomial functions $\tilde{u}_\kappa$, $\tilde{\sigma}_{kl}(k,l=1,2,3)$ along the x_2 and x_3 coordinates for the unknown components of the displacement vector $\mathbf{u}$ and the stress tensor σ:

$$\begin{aligned}&\tilde{u}_1=\sum_{i+j=0}^{M_p}u_1^{(i,j)}x_2^{2i}x_3^{2j+1},\quad \tilde{u}_2=\sum_{i+j=0}^{M_p-1}u_2^{(i,j)}x_2^{2i+1}x_3^{2j+1},\quad \tilde{u}_3=\sum_{i+j=0}^{M_p}u_3^{(i,j)}x_2^{2i}x_3^{2j}\\&\tilde{\sigma}_{11}=\sum_{i+j=0}^{M_p}\sigma_{11}^{(i,j)}x_2^{2i}x_3^{2j+1},\quad \tilde{\sigma}_{23}=\sum_{i+j=0}^{M_p-2}\sigma_{23}^{(i,j)}(a^2-x_2^2)(a^2-x_3^2)x_2^{2i-1}x_3^{2j-2}\\&\tilde{\sigma}_{22}=\sum_{i+j=0}^{M_p}\sigma_{22}^{(i,j)}(a^2-x_2^2)x_2^{2i}x_3^{2j+1},\quad \tilde{\sigma}_{33}=\sum_{i+j=0}^{M_p}\sigma_{33}^{(i,j)}(a^2-x_3^2)x_2^{2i}x_3^{2j+1}\\&\tilde{\sigma}_{12}=\sum_{i+j=0}^{M_p-1}\sigma_{12}^{(i,j)}(a^2-x_2^2)x_2^{2i+1}x_3^{2j+1},\quad \tilde{\sigma}_{13}=\sum_{i+j=0}^{M_p}\sigma_{13}^{(i,j)}(a^2-x_3^2)x_2^{2i}x_3^{2j}\end{aligned}\tag{7.1}$$

where $u_k^{(i,j)}$ and $\sigma_{kl}^{(i,j)}$ are functions of the time t and the x_1 coordinate, defined below.

The proposed approximations satisfy boundary conditions (6.1) exactly on the sides of the solid parallel to the Ox_1 axis. It was shown in Ref. 19 that the symmetry of the solid about the Ox_1x_2 and Ox_1x_3 coordinate planes enables us to split the initial problem into four independent subproblems: three-dimensional tension-compression and torsion along the x_1 axis, and also bending around the x_2 and x_3 axes. For the motions of the cross section $x_1=0$ considered, only bending around the x_2 axis occurs. The proposed functions (6.2) do not disturb the symmetry of the problem.

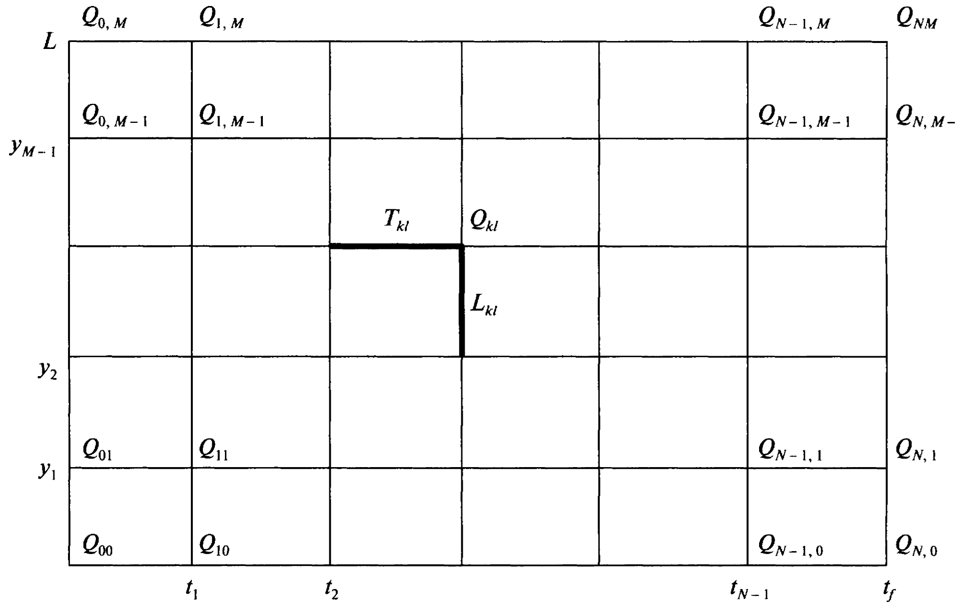

Fig. 2.

We will split the rectangular region $\Upsilon = \{t, x_1 : t \in (0, t_f), x_1 \in (0, L)\}$ in the time space t and coordinate space x_1 into $N \times M$ rectangles Υ_{kl} with vertices $Q_{k-1,\,l-1}$, $Q_{k-1,\,l}$, $Q_{k,\,l-1}$, $Q_{k,l}$ where

$$Q_{k,l} = \{t_k, y_l\}, \quad t_k > t_{k-1}, \quad k = 1,\ldots,N, \quad y_l > y_{l-1}, \quad l = 1,\ldots,M$$
$$t_0 = 0, \quad t_N = t_f, \quad y_0 = 0, \quad y_M = L$$

(see Fig. 2). We will designate the sides of the rectangles Υ_{kl} as

$$L_{kl} = (Q_{k,l-1}, Q_{kl}), \quad k = 0,\ldots,N, \quad l = 1,\ldots,M;$$
$$T_{kl} = (Q_{k-1,l}, Q_{kl}), \quad k = 1,\ldots,N, \quad l = 0,\ldots,M$$

In each four-dimensional region

$$\Sigma_{kl} = \{t, x_1, x_2, x_3 : \{t, x_1\} \in \Upsilon_{kl}, |x_2| < a, |x_3| < a\}$$

we specify the polynomials

$$u_k^{(j_2,j_3)}(t,x_1) = \sum_{j_0+j_1=0}^{N_p} u_k^{(J)} t^{j_0} x_1^{j_1}, \quad \sigma_{kl}^{(j_2,j_3)}(t,x_1) = \sum_{j_0+j_1=0}^{N_p} \sigma_{kl}^{(J)} t^{j_0} x_1^{j_1}; \quad k,l = 1,2,3$$
$$J = \{j_i\}, i = 0,\ldots,5; \quad j_0, j_1 \le N_p, \quad j_2, j_3 \le M_p, \quad j_4 = 0,\ldots,N, \quad j_5 = 0,\ldots,M \tag{7.2}$$

Here $u_k^{(J)}$ and $\sigma_{kl}^{(J)}$ are unknown real coefficients. The integer number N_2 is chosen to be fairly high in order to satisfy the dynamic-equilibrium equation in problem (4.2) and boundary conditions (6.1) and (6.2). When using variational approaches it is necessary, in addition, to satisfy the following joining conditions of the individual elements Σ_{kl}

$$\text{when } \{t, x_1\} \in L_{kl}, \quad k = 1,\ldots,N-1, \quad l = 1,\ldots,M$$
$$\tilde{\mathbf{u}}^{(kl)}(t_k, x_1, x_2, x_3) = \tilde{\mathbf{u}}^{(k+1,l)}(t_k, x_1, x_2, x_3), \quad \frac{\partial \tilde{\mathbf{u}}^{(kl)}}{\partial t}(t_k, x_1, x_2, x_3) = \frac{\partial \tilde{\mathbf{u}}^{(k+1,l)}}{\partial t}(t_k, x_1, x_2, x_3)$$
$$\text{when } \{t, x_1\} \in T_{kl}, \quad k = 1,\ldots,N, \quad l = 1,\ldots,M-1$$
$$\tilde{\mathbf{u}}^{(kl)}(t, y_l, x_2, x_3) = \tilde{\mathbf{u}}^{(k,l+1)}(t, y_l, x_2, x_3), \quad \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}}^{(kl)}(t, y_l, x_2, x_3) = \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}}^{(k,l+1)}(t, y_l, x_2, x_3), \quad \mathbf{n} = (1,0,0) \tag{7.3}$$

8. Numerical results. We will give an example of the use of the algorithm, proposed in Section 5, and we will analyse the numerical results obtained when modelling the motions of an elastic body. We chose the following dimensionless geometrical and mechanical parameters for the calculations

$$L = 1, \quad a = 0.05, \quad t_f = 2, \quad \rho S = 1, \quad EI = 1, \quad \nu = 0.3, \quad S = 4a^2, \quad I = 4a^4/3$$

and we specified the integer numbers defining the order of the approximations of the unknown functions on an element:

$$N = 5, \; M = 1, \; M_p = 2, \; N_p = 10$$

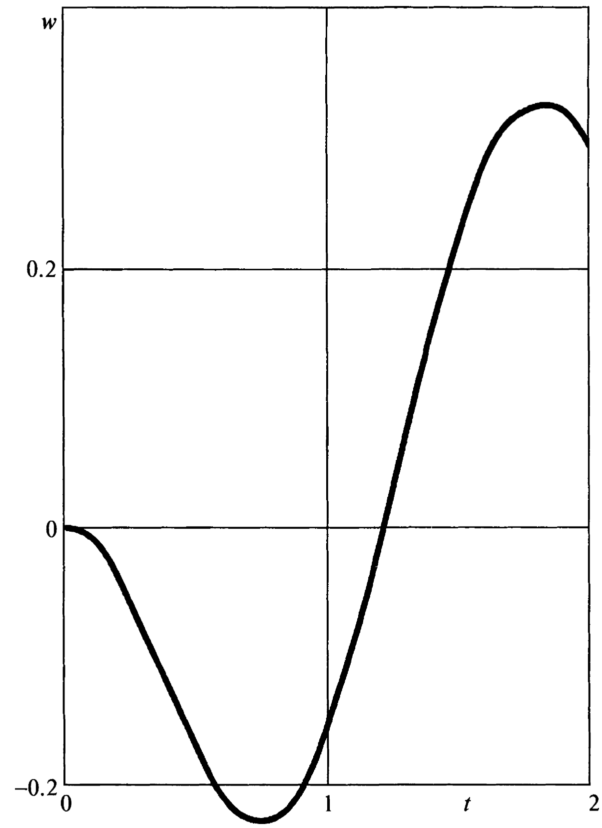

Fig. 3.

We used a mesh uniform in time, with nodal values $t_i = it_f/N$ ($i = 1, \ldots, N$). After satisfying the dynamic-equilibrium relations, and the boundary, initial and interelement conditions, the dimension of the system was $N_{DOF} = 1985$. We chose the control law (6.2), which transfers the section $x_1 = 0$ from the initial zero state to the specified final state, to be cubic ($N_v = 3$) in time, namely,

$$\upsilon = (3t^2 - t^3)/4, \quad \upsilon(0) = \dot{\upsilon}(0) = \dot{\upsilon}(t_f) = 0, \quad \upsilon(t_f) = 1$$

For specified parameters, the value of the integral over time of the total mechanical energy, stored by the body during the motion, is approximately equal to $\psi = 0.1822$. The absolute and relative errors (5.2) were $\Phi_0 = 0.0020$ and $\Delta = 1.1\%$.

In Fig. 3 we show the relative elastic displacement $w = u_3(t, L, 0, 0) - \upsilon(t)$ of the central point of the section $x_1 = L$ as a function of the time t.

The linear total mechanical energy density function of the body

$$\psi_1(t, x_1) = \int_{-a}^{a}\int_{-a}^{a} \psi \, dx_2 dx_3$$

is shown as a function of the time t and the coordinate x_1 in Fig. 4. The local error distribution

$$\varphi_1(t, x_1) = \int_{-a}^{a}\int_{-a}^{a} \varphi_0 \, dx_2 dx_3$$

in sections $x_1 = \text{const}$ along the Ox_1 axis is shown in Fig. 5. It can be seen that the largest error is concentrated about the controlled section $x_1 = 0$.

Unlike the Euler–Bernoulli beam model (see, for example, Ref. 14), the three-dimensional model of the motions of a beam with a square cross section enables us to obtain the local parameters of motion at any point of the elastic body at an arbitrary instant of time. For example, in Fig. 6 we show the shape of the deformed section $x_1 = 0.05$ at the instant of time $t = 0.8$. The distribution of the local error φ_0 in this section at $t = 0.8$ is shown in Fig. 7. It can be seen that the error is close to zero almost everywhere, with the exception of small regions close to the corners of the section. Note that the section was chosen to be close to the fixed end, in order to emphasise the existence of a singularity in the problem. Unfortunately, in order to obtain more reliable solutions in the region of the singularity it is necessary to use different approaches, including methods for the mesh adaptation and mesh refinement, and also the use of singular finite elements. We have used some of these methods to improve the quality of the solution of static problems of the linear theory of elasticity.[20] However, the use of these approaches is outside the scope of this paper.

In the motion considered, the mechanical energy W initially increases from a zero value to a certain maximum value when $t \approx 1.7$, after which it decreases monotonically, as seen from Fig. 8. The local change in the value of the energy W_0 in Fig. 9, due to the discretization of

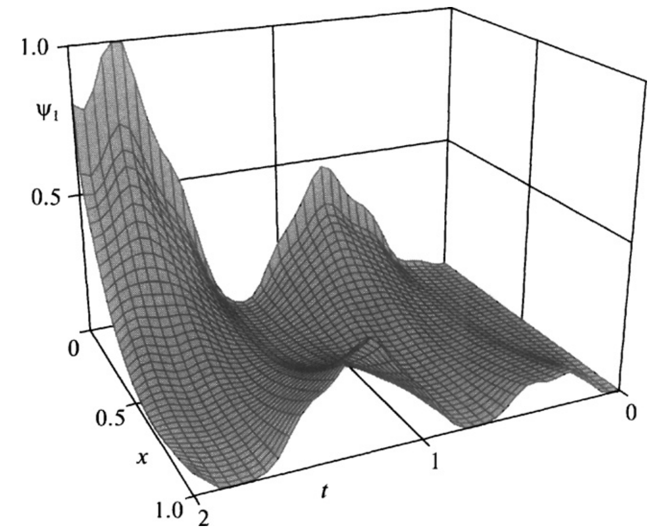

Fig. 4.

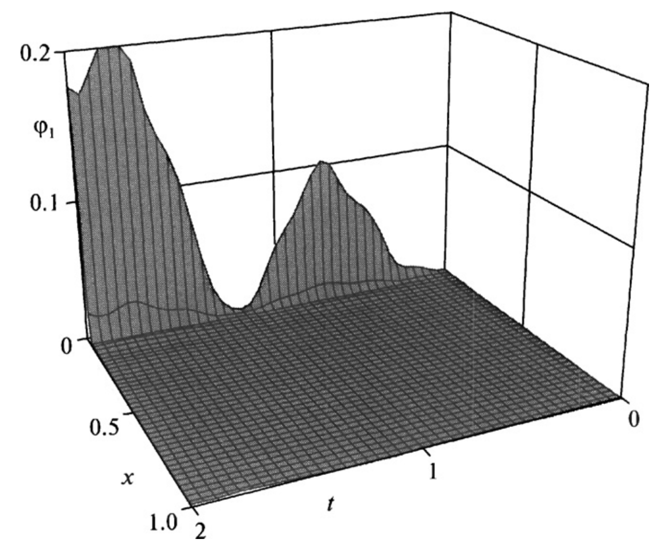

Fig. 5.

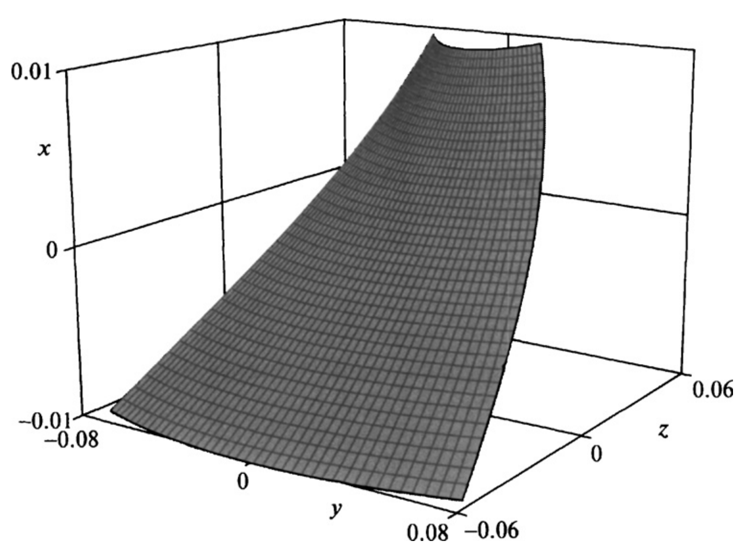

Fig. 6.

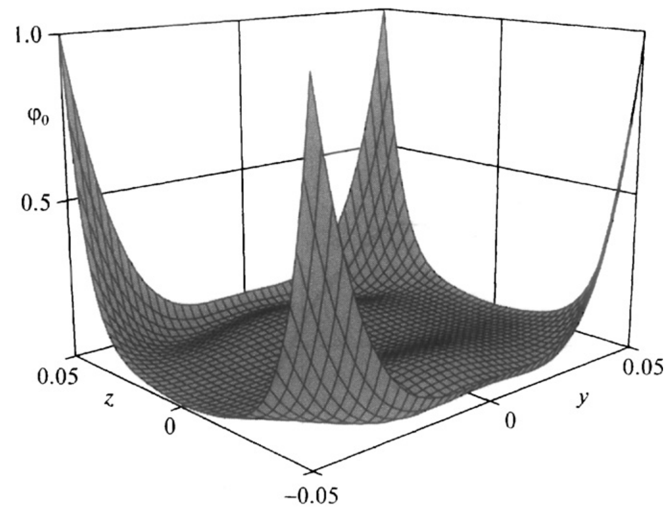

Fig. 7.

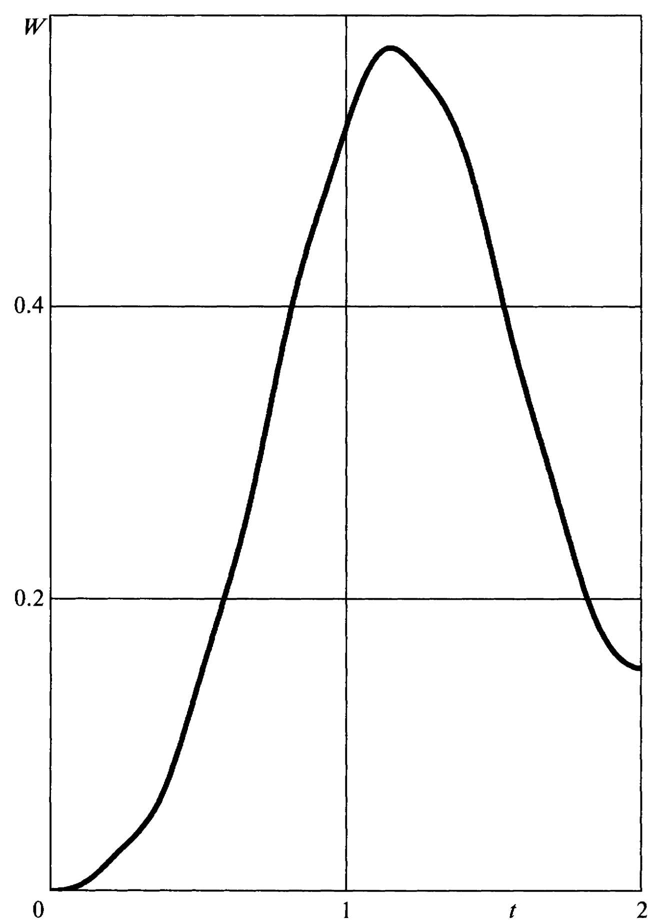

Fig. 8.

the model, reflects the quality of the numerical solution obtained. The value of “parasitic” energy as a function of time is small compared with the total mechanical energy of the beam during motion. When the accuracy of the solution is increased this energy imbalance will be reduced.

The distribution of the linear power density of this “parasitic” energy

$$\omega(t,x_1) = \int_{-a}^{a}\int_{-a}^{a} \tilde{\xi}^* : \mathbf{C} : \frac{\partial \tilde{\varepsilon}^*}{\partial t} dx_2 dx_3$$

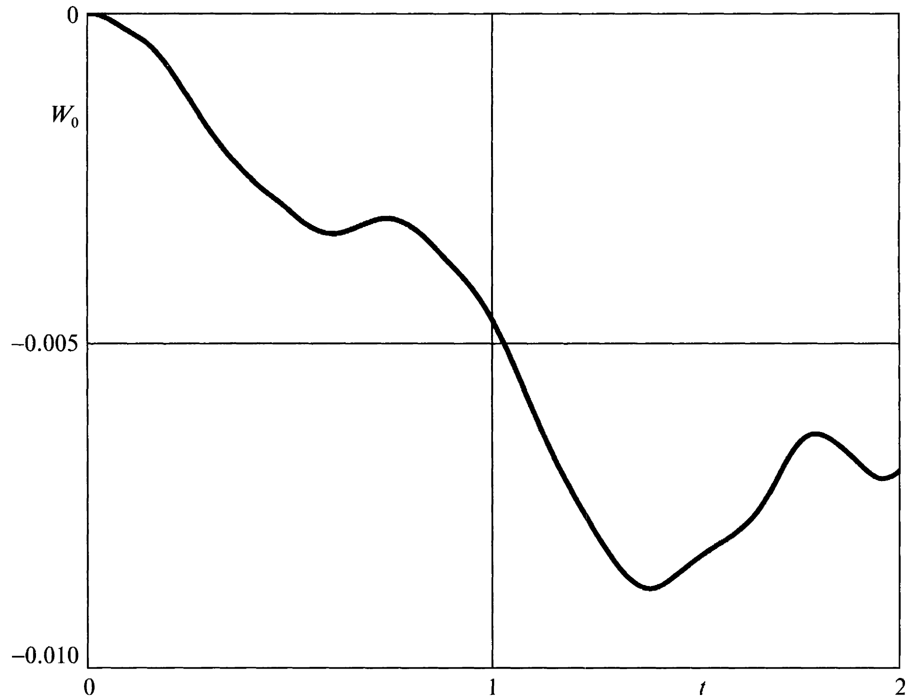

Fig. 9.

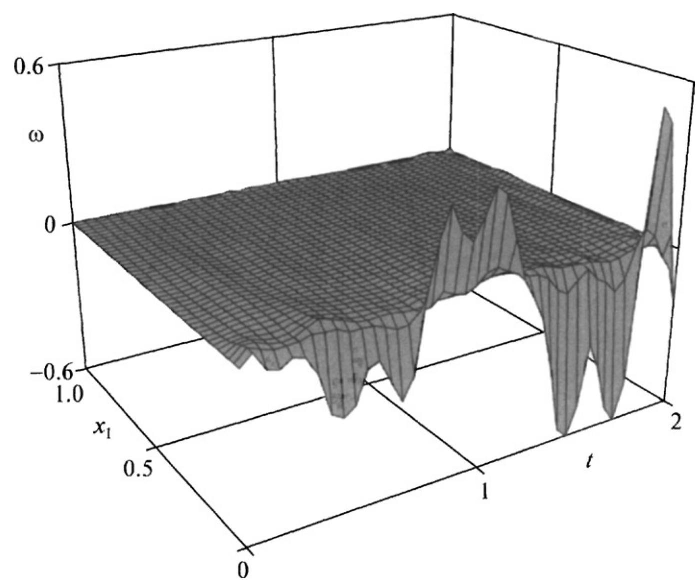

Fig. 10.

is shown in Fig. 10. It can be seen that the maximum change in the energy of a finite-dimensional system occurs close to the controlled section (the fixed end), just where the maximum error in calculating φ_1 is concentrated (see Fig. 5).

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References

1. Washizu K. *Variational methods in elasticity and plasticity*. Oxford: Pergamon Press; 1982.
2. Kravchuk AS, Neittaanmäki PJ. *Variational and quasi-variational inequalities in mechanics*. Dordrecht: Springer; 2007, 329 p.
3. He J-H. Generalized variational principles for thermopiezoelectricity. *Arch Appl Mech* 2002;**72**(4–5):248–56.
4. Atluri SN, Zhu T-L. A new meshless local Petrov–Galerkin (MLPG) approach in computional mechanics. *Comput Mech* 1998;**22**(2). pp. 177–127.
5. Belytschko T, Lu YY, Gu L. Element-free Galerkin method. *Intern J Numer Methods Eng* 1994;**37**(2):229–56.

6. Kwon KC, Park SH, Jiang BN, Youn SK. The least-squares meshfree method for solving linear elastic problems. *Comp Mech* 2003;**30**(3):196–211.
7. Courant R, Hilbert D. *Methoden der Mathematischen Physik, V. 1.*. Berlin: Springer; 1931.
8. Akulenko LD, Kostin GV. The perturbation method in problems of the dynamics of inhomogeneous elastic rods. *Prikl Mat Mekh* 1992;**56**(3):452–64.
9. Akulenko LD, Nesterov SV. *High-Precision Methods in Eigenvalue Problems and their Application.*. Boca Raton etc.: Chaman and Hall/CRC; 2005, 240p.
10. Chernous'ko FL. Decomposition and suboptimal control in dynamical systems. *Prikl Mat Mekh* 1990;**54**(6):883–93.
11. Chernous'ko FL, Ananevskii IM, Reshmin SA. *Methods of Control of Non-linear Mechanical Systems*. Moscow: Fizmatlit; 2006.
12. Leineweber DB, Bauer EI, Bock HG, Schloeder JP. An efficient multiple shooting method based on a reduced SQP strategy for large-scale dynamic process optimization. Pt 1: Theoretical aspects. *Comp and Chem Eng* 2003;**27**:157–66.
13. Kostin GV, Saurin VV. An integrodifferential approach to solving problems of the linear theory of elasticity. *Dokl Ross Akad Nauk* 2005;**404**(5):628–31.
14. Kostin GV, Saurin VV. The construction of controlled motions of an elastic rod by the method of integrodifferential relations. *Izv Ross Akad Nauk Teoriya i Sistemy Upravleniya* 2006;**1**:60–7.
15. Kostin GV, Saurin VV. Optimization of the motions of an elastic rod by the method of integrodifferential relations. *Izv Ross Akad Nauk Teoriya i Sistemy Upravleniya* 2006;**2**:56–64.
16. Kostin GV, Saurin VV. Modelling and optimization of the motions of elastic systems by the method of integrodifferential relations. *Dokl Ross Akad Nauk* 2006;**408**(6):750–3.
17. Kostin GV, Saurin VV. The method of integrodifferental relations for linear elasticity problems. *Arch Appl Mech* 2006;**76**(7–8):391–402.
18. Kostin GV, Saurin VV. Variational formulation of problems of optimizing the motions of elastic bodies. *Dokl Ross Akad Nauk* 2007;**415**(2):180–4.
19. Kostin GV, Saurin VV. An asymptotic approach to the problem of the free oscillations of a beam. *Prikl Mat Mekh* 2007;**71**(4):670–80.
20. Kostin GV, Saurin VV. A variational formulation in fracture mechanics. *Intern J Fructure* 2008;**150**(1–2):195–211.
21. Kostin GV, Saurin VV. Motion analysis and optimization for beam structures. In: Awrejcewicz J, editor. *Modeling, Simulation and Control of Nonlinear Engineering Dynamical Systems: State-of-the-Art, Perspectives and Applications.*. New York: Springer; 2008. p. 201–10.

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